Lecture 7: Castelnuovo's Criterion

Note Title 8/29/2020

Theorem 1. (Castelnuovo's criterion of rationality)

X suface w/ $q = P_2 = 0 \iff X$ is rational.

$q = h^1(X, \mathcal{O}_X)$, $P_2 = h^0(X, \mathcal{O}_X(2K_X))$

Remark: $q = P_1 = 0$ does NOT to conclude rationality.

Enrique surface, which is $\mathbb{Z}_2$ quotient of a K3 surface.

Definition: A variety X of dimension n is <u>unirational</u>

if $\exists\ \mathbb{P}^n \dashrightarrow X$ <u>dominant</u>.

generic surjective

ex. Every unirational curve is rational.

Corollary: Every unirational surface rational.

pf:

$$\begin{tikzcd} & S \arrow[dl, "\text{composition of blow-ups}"'] \arrow[dr, "f"] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & X \end{tikzcd}$$

$q(S) = P_2(S) = 0$

If $q(X) = h^0(X, \Omega_X) \neq 0$ or $P_2(X) = h^0(X, 2K_X) \neq 0$,

pull back to S implies $q(S) \neq 0$ or $P_2(S) \neq 0$

Remark: (Clemens-Griffiths)

Smooth cubic 3-fold are unirational but not rational.

Proposition 1: X minimal surface w/ $q = P_2 = 0$,

then X contains a smooth rational curve C, $C^2 \geq 0$.

proof of Theorem 1:

WLOG may assume X is minimal.

Proposition 1 $\Rightarrow$ $\exists\, C$ smooth rat'l curve. $C^2 \geq 0$

Ruled surfaces/$\mathbb{P}^1$ are rational.

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$$

$$\Rightarrow \quad 0 \to \underset{\substack{\| \\ 1}}{H^0(X,\mathcal{O}_X)} \to H^0(X,\mathcal{O}_X(C)) \to \underset{\substack{\rotatebox{90}{$\leq$}\ \text{R.R} \\ C^2-0+1}}{H^0(C,\mathcal{O}_C(C))} \to \underset{\substack{\| \\ 0}}{H^1(X,\mathcal{O}_X)} \ \because q=0$$

$$\Rightarrow h^0(X,\mathcal{O}_X(C)) \geq C^2+2 \geq 2$$

Choose $\underset{C^*}{D} \in |C|$ s.t. $C \cap D$ finitely many points.

$$\begin{tikzcd} \tilde{X} = \text{blow up base points of } |C,D| \arrow[d] \arrow[r, "|C,D|"] & \mathbb{P}^1 \\ X \arrow[ur, dashed] & \end{tikzcd}$$

ruled surface thus rational.

$\square$

Lemma 1. X minimal surface w/ $K^2 < 0$

$\forall\, a > 0$, $\exists\, D \geq 0$ s.t. $K.D \leq -a$, $|K+D| = \phi$.

pf: • It suffices to prove $\exists\, E \geq 0$, $K.E < 0$

Take C be a component of E w/ $K.C < 0$

If $C^2 < 0$, then $-2 \leq \deg K_C = (K+C).C \leq -2$

$$\Downarrow$$

"=" holds, $K.C = -1$, $C^2 = -1$

C is a (-1)-curve $\rightarrow\!\!\!\!\times$

Thus $C^2 \geq 0$.

$(nK + aC).K < -a$, $\forall\, n \geq 0$

$(nK+aC).C<0$ as $n \gg 0$

$\exists n \in \mathbb{N}$ s.t $|nK+aC| \neq \phi$, $|(n+1)K+aC| = \phi$

Then one can take $D = nK+aC$.

- H: hyperplane section first choice to produce an effective divisor!

① $H.K<0$, then can take $E=H$.

② $H.K=0$, then $\underbrace{|nH+K|}_{E} \neq \phi$, for $n \gg 0$

③ $H.K>0$, $(H+r_0K).K=0$, $r_0 = -\frac{H.K}{K^2}$

Choose $a,b \in \mathbb{N}$ s.t $\frac{b}{a} \sim r_0$, $\frac{b}{a} > r_0$

Take $E = aH+bK$. $(aH+bK).K<0$

$$h^0(E) + \underbrace{h^0(-aH-(b-1)K)}_{=0} \geq \chi(\mathcal{O}_X) + \frac{1}{2}(aH+bK).(aH+(b-1)K) \gg 0$$

$(aH+bK).H>0$ $a,b \gg 0$

$(-aH-(b-1)K).H<0$ $(aH+bK).K \sim 0$

$\square$

Proof of Proposition 1.

It suffices to prove that $\exists D \geq 0$, $D.K<0$, $|K+D| = \phi$

C component of D s.t $C.K<0$, $|K+C| = \phi$

then $0 = h^0(-C) + h^0(K+C) \geq \chi(\mathcal{O}_X) + \frac{1}{2}(-C).(-C-K)$

$\overset{q=p_2=0}{=} 1 + \frac{1}{2}(K+C).C = g_C$

$g_C = 0$, $C^2<0$ only when $K.C = C^2 = -1$ $\rightarrow\!\!\times$

Thus, $C^2 \geq 0$

- $K^2 < 0 \xrightarrow[\text{Lemma 1}]{} $ Proposition 1

- $K^2 = 0$, $\exists n \in \mathbb{N}$ s.t $|H+nK| \neq \phi$, $|H+(n+1)K| = \phi$

  then can take $D \in |H+nK|$, then apply Lemma 1.

  $$\underbrace{h^0(-K)+h^0(2K)}_{h^0(2K)=0} \geq \underbrace{\chi(\mathcal{O}_X)}_{=1} + \tfrac{1}{2}(-K)(-K-K) = 1 \quad \therefore -K.H > 0$$

- $K^2 > 0$, then $h^0(-K) \geq 2$

  If $\exists A+B \in |-K|$, $A, B \geq 0$, then $|K+A| = |-B| = \phi$

  Take $D = A$ and apply Lemma 1.

  Therefore may assume that elements in $|-K|$ are irreducible.

  Take any effective divisor H

  $n \in \mathbb{N}$ s.t $|H+nK| \neq \phi$, $|H+(n+1)K| = \phi$

① If $H+nK \not\equiv 0$, $E = \sum n_i \underline{C_i} \in |H+nK|$, $n_i \geq 0$

irreducible

then $|K+C_i| = \phi$

$$|-K| \neq \phi \Rightarrow -K.E \geq 0 \quad \text{or} \quad \underline{K.E \leq 0}$$

$$\Downarrow$$

$$\exists C_i,\ K.C_i \leq 0$$

i) $C_i^2 \geq 0$ then the Lemma is proved

ii) $C_i^2 = -1$ contradicts to minimality of X.

iii) Otherwise, $-2 \leq (K+C_i).C_i \Rightarrow K.C_i = 0$, $C_i^2 = -2$

Claim: $h^0(-K-C_i) > 0$ then $\exists A \in |-K-C_i|$, $A + C_i = -K$ ✗

$$h^0(-K-C_i) + \underbrace{h^0(2K+C_i)}_{=0} \geq 1 + \tfrac{1}{2}(-K-C_i).(-2K-C_i)$$

$$= 1 + \tfrac{1}{2}(2K^2 + 3K.C_i + C_i^2) \geq 1$$

$$h^0(2K+C_i) \leq h^0(K+C_i) = 0$$

$\because |K| \neq \phi$ Otherwise, $(-K).(K+C_i) \geq 0$

$$\overset{\|}{-K^2 + K.C_i} \quad \cancel{\;}$$

$$\overset{\wedge}{0}$$

② All the effective divisor $H \equiv nK$, for some $n \in \mathbb{Z}$.

$$\underset{\overset{\|}{0}}{H^1(X,\mathcal{O}_X)} \longrightarrow \underset{\overset{\wr\|}{\mathbb{Z}}}{H^1(X,\mathcal{O}_X^*)} \xrightarrow{\cong} H^2(X,\mathbb{Z}) \longrightarrow \underset{\overset{\|}{0}}{H^2(X,\mathcal{O})} \qquad \therefore b_2 = 1$$

Poincaré duality $\Longrightarrow K^2 = 1$

Noether's formula $\chi(\mathcal{O}_X) = \frac{1}{12}\left(K^2 + \underline{\chi_{top}(X)}\right)$

$$1 = \frac{1}{12}\left(1 + \underline{2-2b_1+b_2}\right) \Longrightarrow b_1 = -4 \quad \cancel{\;}$$

$\square$

Theorem 2. X = minimal rational surface then $X \cong \mathbb{P}^2$ or $\mathbb{F}_n$, $n \neq 1$.

pf: $A = \{ C^2 \mid C$ smooth rat'l curve in X, $C^2 \geq 0 \} \neq \phi$ Proposition 1

w/ minimum element m.

C: smooth rational curve w/ • $C^2 = m$,

• minimize $H.C$

① All elements in $\underline{|C| \text{ are smooth rational curves.}}$

$\Downarrow$ adjunction

$D \in |C|$, $D = \sum_i n_i C_i$ $\quad (K+C).C = -2$

$$H^0(X,\mathcal{O}_X) \twoheadrightarrow H^0(C,\mathcal{O}_C) \longrightarrow H^1(X,\mathcal{O}_X(-C)) \longrightarrow \underset{\overset{\|}{0}}{H^1(X,\mathcal{O}_X)} \Longrightarrow H^1(X,\mathcal{O}_X(-C)) = 0$$

R.R $\quad h^0(K+C) - \underset{\overset{\|}{0}}{h^1(-C)} + \underset{\overset{\|}{0}}{h^0(-C)} = 1 + \frac{1}{2}(K+C).C = 0$

$\Longrightarrow \quad H^0(K+D) = H^0(K+C) = 0$

$\Longrightarrow \quad H^0(K+C_i) = 0$,

Replace C by C_i in above long exact sequence $\Longrightarrow g(C_i)=0$

$K.D = \underline{K.C < 0} \Longrightarrow K.C_i < 0$ for some i.

$\because C^2 \geq 0,\ K.C + C^2 = -2$

$-2 = (K+C_i).C_i \Longrightarrow C_i^2 \geq 0$

$C_i^2 \neq -1$

$m = \underline{C^2 \geq n_i^2 C_i^2} \longrightarrow n_i = 1,\quad C_i^2 = m$

C^2 minimal

$C^2 = D^2 = (D' + n_i C_i)^2$

$= n_i^2 C_i^2 + n_i \underset{\overset{\vee\!|}{0}}{\underline{D'.C_i}} + \underline{D.D'}$

$\overset{\|}{C.D'}$

$\overset{\vee\!|}{0}$

$H.C_i \leq H.C$

$\longrightarrow \quad n_{j \neq i} = 0 \quad D = C_i$ smooth rat'l.

$H.C$ minimal.

② $\dim|C| < 3$

A double point is codim 3 in the linear system.

If $\dim|C| \geq 3$, it would contradicts to ①

③ $|C|$ is base-point free.

$C_0 \in |C|$. $\quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_{C_0}(m) \longrightarrow 0$

$0 \to H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X(C)) \twoheadrightarrow H^0(C_0, \mathcal{O}_{C_0}(m)) \longrightarrow H^1(X, \mathcal{O}_X)$

$1 \qquad\qquad m+1 \qquad\qquad H^1(X,\mathcal{O}_X) \overset{\|}{=} 0 \quad q=0$

$h^0(C_0, \mathcal{O}_{C_0}(m)) - h^1(C_0, \mathcal{O}_{C_0}(m)) = m - 0 + 1$

$h^1(C_0, \mathcal{O}_{C_0}(m)) = h^0(C_0, \mathcal{O}_{C_0}(-2-m)) = 0$

$\Longrightarrow \quad h^0(X, \mathcal{O}_X(C)) = m+2$

$\mathcal{O}_{\mathbb{P}^1}(m)$ is base point free

$\overset{\vee\!|}{0}$

So $|C|$ has no base point on C_0.

④ ② + ③ $\Rightarrow$ $m+2 \le 3$ $\therefore$ $m = 0$ or 1

$m=0$, $X \xrightarrow{|C|} \mathbb{P}^1$ minimal ruled surface

$\therefore$ $X \cong \mathbb{P}(E) \longrightarrow \mathbb{P}^1$, E rank 2

Grothendieck's thm $E \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$

$\therefore$ $X \cong \mathbb{F}_n$, for some $n \in \mathbb{N}$

$\mathbb{F}_1 \cong Bl_p \mathbb{P}^2$ not minimal.

$m = 1$, $X \xrightarrow{|C|} \mathbb{P}^2$

1:1 thus $\cong$

$\square$

Theorem 3. X, X' non-ruled minimal surfaces

$f: X' \longrightarrow X$ birational morphism, then $f: X' \xrightarrow{\cong} X$.

In particular, non-ruled minimal surface is unique up to isomorphisms.

We will need to introduce the technology of Albanese map.

Theorem 4. X = smooth variety

$\exists$ abelian variety A, $\alpha: X \longrightarrow A$ morphism

w/ the universal property:

Given

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\alpha"'] & T \\ & A \arrow[u, "\exists ! \tilde{f}"'] \end{tikzcd}$$

complex torus

$\exists ! \tilde{f}$ s.t. $f = \tilde{f} \circ \alpha$

generation of Jacobians of Riemann surfaces

A is unique, up to isomorphism and $H^0(A,\Omega^1_A) \overset{\alpha^*}{\cong} H^0(X,\Omega^1_X)$

$\parallel$

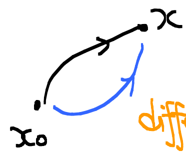
$\mathrm{Alb}(X)$ Albanese variety of X

pf: Fix $x_0 \in X$, choose $\omega_1, \cdots, \omega_N$ basis of $H^0(X,\Omega^1_X)$

$N = h^0(X,\Omega^1_X)$

Define
$$X \longrightarrow \mathbb{C}^N \cong H^0(X,\Omega^1_X)^*$$
$$x \longmapsto \left(\int_{x_0}^{x} \omega_1, \cdots, \int_{x_0}^{x} \omega_N\right)$$

same path from x_0 to x.



difference is in $H_1(X,\mathbb{Z})$

holo. forms on Kähler manifold are d-closed

$\overline{\partial}^* \alpha = 0$ by type reason

thus harmonic.

$$\rightsquigarrow \quad X \xrightarrow{\alpha} \mathbb{C}^N / \mathrm{Im}\left(H_1(X,\mathbb{Z}) \to H^0(X,\Omega^1_X)^*\right) =: A$$

- α holomorphic $\quad \overline{\partial} \int_{x_0}^{x} \omega_i = \left(d \int_{x_0}^{x} \omega_i\right)^{0,1} \underset{FTC}{=} (\omega_i)^{0,1} = 0$

- A abelian variety

  need to check Rieman bilinear relation.

  $\uparrow$

  Hodge theory

- $H^0(A,\Omega^1_A) \xrightarrow[\cong]{\alpha^*} H^0(X,\Omega^1_X)$

  If $A = V/\Lambda$ (vector space V, lattice Λ), then $H^0(A,\Omega_A) \cong V^*$

  $H_1(A,\mathbb{Z}) \cong \Lambda$

- Different choice of $p \rightsquigarrow$ translation of A by lattice.

  $\omega_1, \cdots, \omega_N \rightsquigarrow$ linear auto. of A.

Given $X \xrightarrow{f} T$, T: complex torus

$$\rightsquigarrow \quad H^0(T,\Omega^1_T) \xrightarrow{f^*} H^0(X,\Omega^1_X) \overset{(\alpha^*)^{-1}}{\cong} H^0(A,\Omega^1_A)$$

dually $\rightsquigarrow$

$$\begin{tikzcd}
H^0(T,\Omega^1_T)^* & H^0(X,\Omega^1_X)^* \arrow[l, "f_*"'] & H^0(A,\Omega^1_A)^* \arrow[l, "\cong"', "(\alpha_*)^{-1}"] \\
& H_1(X,\mathbb{Z})/_{tor} \arrow[u] & \\
H_1(T,\mathbb{Z}) \arrow[uu, phantom, "\cup\!\mathrm{I}"] & H_1(X,\mathbb{Z}) \arrow[l, "f_*"'] \arrow[u] \arrow[uu] & H_1(A,\mathbb{Z}) \arrow[l] \arrow[ul, "\cong"', "(\alpha_*)^{-1}"] \arrow[uu, phantom, "\cup\!\mathrm{I}"]
\end{tikzcd}$$

X projective $\quad H_1(X,\mathbb{C})^* \cong H^1(X,\mathbb{C}) \cong H^{1,0} \oplus H^{0,1}$

$\therefore \ H_1(X,\mathbb{C}) \times H^{1,0} \longrightarrow \mathbb{C}$ non-degenerate

$$\longrightarrow \quad A = H^0(A,\Omega^1_A)^* \big/ H_1(A,\mathbb{Z}) \xrightarrow{\tilde{f}} H^0(T,\Omega^1_T)^* \big/ H_1(T,\mathbb{Z}) \cong T$$

$\square$

Remark: • (Functoriality)

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\alpha_Y"] \\
\mathrm{Alb}(X) \arrow[r, "\alpha_f"] & \mathrm{Alb}(Y)
\end{tikzcd}$$

• If $H^0(X,\Omega^1_X)=0$, then $X \longrightarrow T$ can only be constant.

• $\mathrm{Alb}(X)$ is generated by $\alpha(X)$.

the abelian subvariety generated by $\alpha(X)$ satisfies the universal property.

Proposition 2. X surface, $\alpha : X \longrightarrow \mathrm{Alb}(X)$ Albanese map. Assume that $\alpha(X)=C$ is a curve. Then C is a smooth curve of genus $q=h^1(X,\mathcal{O}_X)$. Moreover, $\alpha : X \longrightarrow C$ has connected fibres.

Lemma 2: Assume factorization

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "\alpha"'] & Y \arrow[r, "j"] & \mathrm{Alb}(X)
\end{tikzcd}$$

. f surjective.

Then $\tilde{j} : \mathrm{Alb}(Y) \xrightarrow{\cong} \mathrm{Alb}(X)$.

pf:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[drr, bend left, "\alpha"] & Y \arrow[d, "\alpha'"] \arrow[dr, "j"] & \\
\mathrm{Alb}(X) \arrow[r, "F"] & \mathrm{Alb}(Y) \arrow[r, "\tilde{j}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Alb}(X) \arrow[d, "F"] \\
& & \mathrm{Alb}(Y)
\end{tikzcd}$$

universal property

$\Longrightarrow$ $\tilde{j} \circ F = \mathrm{id}$

$F \circ \tilde{j} = \mathrm{id}$

$\square$

Proof of Proposition 2:

$$\begin{tikzcd}
& \tilde{C} \arrow[dr] & & \\
X \arrow[ur] \arrow[r, two heads] \arrow[rr, bend right, "\alpha"'] & C \arrow[r] & \mathrm{Alb}(X)
\end{tikzcd}$$

$\tilde{C}$ normalization of C

$\overset{\text{Lemma 2}}{\Longrightarrow}$

$$\begin{tikzcd}
\tilde{C} \arrow[d] \\
\mathrm{Alb}(X) \cong \mathrm{Alb}(\tilde{C}) \cong \mathrm{Jac}(\tilde{C})
\end{tikzcd}$$

$\Rightarrow$ C smooth

$\therefore\ g(C) = \dim \mathrm{Alb}(X)$

$= h^0(X, \Omega^1_X)$

(Stein factorization) $f: X \to Y$ proper morphism

then $\exists!$ $f: X \xrightarrow[\text{fibres}]{\text{connected}} \tilde{Y} \xrightarrow[\text{morphism}]{\text{finite}} Y$.

Stein factorization $\Longrightarrow$

$$\begin{tikzcd}
X \arrow[r, "\text{connected fibres}"'] \arrow[dr] & C' \arrow[r, "\text{finite}"] & C \arrow[r] & \mathrm{Alb}(X) \\
& \tilde{C}' \arrow[u] & &
\end{tikzcd}$$

$\tilde{C}'$ normalization of C'

Lemma 2 $\Longrightarrow$ $\mathrm{Alb}(\tilde{C}') = \mathrm{Alb}(C)$ $\therefore$

$$\begin{tikzcd}
\tilde{C}' \arrow[rr, "\cong"] \arrow[dr] & & C \\
& C' \arrow[ur] &
\end{tikzcd}$$

Proposition 3. X surface w/ $\underset{h^2(X,\mathcal{O}_X)}{p_g} = 0$, $\underset{h^1(X,\mathcal{O}_X)}{q} \geq 1$,

then $\alpha(X) \subseteq \mathrm{Alb}(X)$ is a curve.

pf: If $\alpha(X)$ is a surface, then

$$\begin{tikzcd} X \arrow[r, "\alpha"] & \alpha(X) \\ \tilde{U} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & U \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

$X \xrightarrow{\alpha} \alpha(X)$ is generic finite

$\tilde{U} \to U$ étale open

$$U = \{z_3 = z_4 = \cdots = z_N = 0\} \subseteq \mathrm{Alb}(X)_{(z_1, \cdots, z_N)}$$

$$\therefore \quad \omega = dz_1 \wedge dz_2 \in H^0(\mathrm{Alb}(X), \Omega^2_{\mathrm{Alb}(X)})$$

$$\begin{tikzcd} \omega \arrow[d, mapsto] & H^0(\mathrm{Alb}(X), \Omega^2_{\mathrm{Alb}(X)}) \arrow[d] \\ \text{nonzero element} \in & H^0(\alpha(X), \omega_{\alpha(X)}) \arrow[d] \\ & H^0(U, \omega_U) \end{tikzcd}$$

$$\underset{\substack{\neq \\ 0}}{\alpha^*\omega} \in H^0(X, \Omega^2_X)$$ contradicts to $h^2(X, \mathcal{O}_X) = 0$.

□

Proof of Theorem 3:

$$\begin{tikzcd} & \exists X'' \arrow[dl, "\pi"'] \arrow[d, dotted, "\varepsilon_n \cdots \varepsilon_1"] \\ X \arrow[r] & X' \end{tikzcd}$$

sequence of blow up

birational morphism

E = exceptional divisor of ε_n

WLOG, assume that $\pi(E) = C$ is a curve

Otherwise π will be factored through ε_n.

Recall that $X'' \xrightarrow{\pi} X$ is also a sequence of blow-ups.

For each blow-up,

$$\begin{tikzcd} E \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & P \arrow[d, phantom, sloped, "\in"] \\ \tilde{Y} \arrow[r] & Y \\ \pi^{-1}D \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & D \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

$m = \mathrm{mult}_p(D)$

$$K_{\tilde{Y}}\cdot\pi^{-1}D = (\pi^* K_Y + E)\cdot(\pi^* D - mE) = K_Y\cdot D + m \underset{\sim}{\geq} K_Y\cdot D$$

$=$ if $D \not\ni p$

Therefore, $\quad -1 = K_{X''}\cdot\pi^{-1}D \underset{\sim}{\geq} K_X\cdot C$

$=$ holds if E is disjoint from the proper transform of exceptional curves of $X'' \to X$.
but then $C^2=-1$ rational curve
contradicts to X is minimal.

$$-2 \leq (K_X + C)\cdot C = \underset{\underset{-2}{\wedge\!|}}{K_X\cdot C} + C^2 \implies C^2 \geq 0$$

$$\boxed{K_X\cdot C \leq -2,\ C^2 \geq 0 \implies P_n(X) = 0}$$

Otherwise, $\exists D \in |nK_X|$, $D\cdot C \geq 0$
$\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad nK_X\cdot C < 0$

- $q=0$, Castelnuovo's criterion $\implies$ X is rational $\not\longrightarrow$
  X non-ruled, $\mathbb{P}^2 \not\to \mathbb{F}_{n\neq 1}$

- $q \geq 1$, Proposition 3 $\implies$

$$\begin{tikzcd} C \subseteq X \arrow[r, "\alpha"] & \alpha(X) \subseteq \mathrm{Alb}(X) \\ E \subseteq X'' \arrow[u] & \end{tikzcd}$$

of genus $q > 1$

$P_g(X) = 0$, $q(X) \geq 1$
$\implies \alpha(X) \subseteq \mathrm{Alb}(X)$
curve of genus q.

C rational $\implies$ C contains in fibre of α.

intersection matrix of fibre components negative definite $\implies$ $\alpha^{-1}(x) = rC \implies C^2 = 0$

$r = 1$

$\implies$ X is ruled surface
(Noether-Enrique)

Theorem 4. X surface w/ $-K_X$ ample
then X is a del Pezzo surface.

pf: • X is rational i.e $h^1(X,\mathcal{O}_X) = h^2(X,\mathcal{O}_X) = 0$

$h^2(X,\mathcal{O}_X) = h^0(X, 2K_X) = 0$ ∵ $-K$ ample

$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \chi_{top}(X)) = \frac{1}{12}(\underset{\vee \atop 0}{K_X^2} + 2 - 2b_1 + \underset{\vee \atop 0}{b_2})$

$\underset{\shortparallel \atop 1}{h^0(X,\mathcal{O}_X)} - h^1(X,\mathcal{O}_X) + \cancel{h^2(X,\mathcal{O}_X)}^{\,0}$ $\Rightarrow 1 - q \geq \frac{1}{12}(1 + 2 - 4q + 1)$ ∴ $q = 0$ or 1

$q = 1 \xRightarrow[P_g = 0 \,+\, \text{Proposition 3}]{} X \longrightarrow \overset{B \atop \shortparallel}{\alpha(X)} \subseteq Alb(X)$
elliptic curve

F : irreducible

adjunction $\Rightarrow 2g_F - 2 = \underset{\wedge \atop 0}{K_X.F} + \underset{\shortparallel \atop 0}{F^2} \Rightarrow g_F = 0$
$K_X.F = -2$

Claim: Fibres of α are all smooth rational curves

$\Rightarrow X$ is a ruled surface over an elliptic curve

$K_X^2 = 8(1 - g_{\alpha(X)}) = 0 \quad \rightarrow\!\!\!\!\!\times$

Otherwise, there exists a fibre $F' = \sum n_i C_i$, $n_i > 0$, C_i: irreducible

$-K_X$ ample $\Rightarrow$ ① $F' = C_1 + C_2$ or ② $F' = 2C$

$K_X . F' = -2$

① $F = C_1 + C_2 \Rightarrow K_X . C_i = -1$

$0 = (C_1 + C_2)^2 = C_1^2 + \underset{\vee \atop 0}{2C_1.C_2} + C_2^2$ Proposition 2 ∴ one of $C_i^2 < 0$, say $C_1^2 < 0$

adjunction $\Rightarrow C_1$ (-1)-curve

$(C_1 + C_2)^2 = 0 \Rightarrow C_2$ (-1)-curve

C_i is a (-1)-curve

$$\begin{array}{c} X \\ \pi \downarrow \\ X' \end{array} \quad \text{blow down } C_i$$

then $K_{X'}^2 = K_X^2 + 1 > K_X^2 > 0$

② $F' = 2C$, then C is a (-1)-curve by adjunction.

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, "\pi"'] & B \\ X' \arrow[ru, two heads] & \end{tikzcd}$$

blow down C

surjective morphism to a smooth curve

thus is flat

fibres are equi-dimensional $\cancel{\ \ }$

Therefor, $q = 0$,

then apply Castelnuovo's criterion.

- Assume that X_{min} is the minimal model of X.

Theorem 2 $\Rightarrow$ $X_{min} \cong \mathbb{P}^2$ or $\mathbb{F}_{n \neq 1}$

If $X_{min} \cong \underline{\mathbb{F}_{n \geq 2}}$, then $-K_X$ is not ample.

$-K_{\mathbb{F}_{n\geq 2}}$ not ample

- $X_{min} \cong \mathbb{P}^2$, then $X \longrightarrow X_{min}$ blow up at r points

$0 \leq r \leq 8$

$X_{min} \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $X \cong X_{min}$

$\Big\}$ otherwise $-K_X$ is not ample

$\square$